Algebraic Number Theory
Semestral examination 2019
M.Math. II - Instructor — B.Sury
Answer any FIVE questions
Maximum marks 50
A score higher than 50 will be equated to 50.

Q 1. (2+2+3+3 marks)
(i) Let K be a number field of degree n over $\mathbb{Q}$. Do there exist elements $a_1, \cdots, a_n \in O_K$ such that the $disc(a_1, \cdots, a_n) = -disc(K)$? Give reasons.
(ii) Let $K = \mathbb{Q}(\theta)$ where $\theta^3 - 6\theta + 36 = 0$. Show that $\theta^2/6 \in O_K$.
(iii) In $\mathbb{Q}(\sqrt{-5})$, find two elements generating the fractional ideal $(3, 1 + 2\sqrt{-5})^{-1}$.
(iv) Let $K = \mathbb{Q}(\sqrt{p})$ where p os a prime congruent to 5 or 7 mod 8. Prove that there is no element of norm -2 in K.
Hint. You may use quadratic reciprocity law.

Q 2. (2+3+3+4 marks)
(i) Prove that the domain $\mathbb{Z} + \mathbb{Z}\sqrt{2} + \mathbb{Z}\sqrt{5} + \mathbb{Z}\sqrt{10}$ is not integrally closed in its quotient field.
(ii) If d is a square-free positive integer and $p \equiv 3$ mod 4 is a prime dividing d, then show that the fundamental unit of $\mathbb{Q}(\sqrt{d})$ must have norm 1.
(iii) Let p be an odd prime and $\zeta = e^{2i\pi/p}$. For $1 \leq r < p$, consider the real numbers

$$t_r = \frac{(1-\zeta^r)(1-\zeta^{-r})}{(1-\zeta)(1-\zeta^{-1}}.$$

Prove that $\sqrt{t_r}$ are real units in $\mathbb{Q}(\zeta)$.
(iv) Recall that an ideal I is said to be primary if $ab \in I$, $a \notin I$ implies $b^n \in I$ for some $n > 0$. If I is an integral ideal which is primary in a Dedekind domain, prove that I is the power of a prime ideal.

Q 3. (10 marks)
Let K be a number field and let p be a prime number. If $pO_K = P_1^{e_1} \cdots P_g^{e_g}$ for prime ideals P_i in O_K, then prove $P_1, \cdots, P_g$ are the prime ideals lying over p.

OR

Consider $K = \mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity. Show that a prime $p \in \mathbf{Z}$ splits completely in $\mathcal{O}_K$ if, and only if, $p \equiv 1 \bmod n$.

Q 4. (11 marks)
For $K = \mathbb{Q}(\sqrt[3]{2})$, prove $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{2}]$.
Hint. You may use the fact that for a field $L = \mathbb{Q}[\alpha]$ for $\alpha \in O_L$, if the minimal polynomial of α is p-Eisenstein for a prime p, then p does not divide $|O_L/\mathbb{Z}[\alpha]|$.

OR

Prove that for any Galois extension K of $\mathbf{Q}$, the Galois group is generated by the various inertia subgroups of the primes.
Hint: Look at the fixed field E of the subgroup generated by all inertia subgroups and show that no primes can ramify in it; then use the fact $disc(E) > 1$ if $E \neq \mathbb{Q}$.

Q 5. (8 marks)
Let $[L : K] = n$ where K is a field K which is complete with respect to a non-archimedean absolute value $|.|_K$. Prove that $|x|_L := |N_{L/K}(x)|^{1/n}$ defines a non-archimedean absolute value on L that extends the one on K.

OR

Let $|.|_K$ be a non-archimedean absolute value on a field K. Let $(L, |.|_L)$ be the completion of K with respect to $|.|_K$. Prove that for each $x \in L$, there exists $y \in K$ such that $|x|_L = |y|_K$.

Q 6. (11 marks)
For a modulus m of a number field K, define I^m, K_m and $K_{m,1}$. Prove that the ray class group $I^m/i(K_{m,1})$ is finite, of order a multiple of the class number.
Hint. You may use the finiteness of $K_m/K_{m,1}$ and that each coset of $K_{m,1}$ in K_m contains an element not divisible by any given ideal.

OR

Obtain the order of the group of roots of unity in $\mathbf{Q}_p$ for some odd prime p. Deduce that $\mathbf{Q}_p$ is not isomorphic to $\mathbf{Q}_q$ for odd primes $p \neq q$.

Q 7. (12 marks)
Let L_1, L_2 be Galois extensions of a number field K. Show that if the set of primes of K which split completely in L_1 is the same as the set of primes splitting completely in L_2 (except for a set of density zero), then $L_1 = L_2$.
Hint. Use Frobenius's density theorem to find the densities of primes splitting completely in L_1L_2.

OR

Let m be a modulus for a number field K. Let H be a subgroup of I^m containing $i(K_{m,1})$. Prove that if S is any set of primes in H which has a density $\delta(S)$, then this density is at most $1/[I^m : H]$.
Hint. You may use the fact that $(s-1)\zeta_K(s)$ has a limit as s tends to 1 from the right.